INVESTIGACION

La sección de Difusión de Investigación en Ingeniería, como su nombre lo indica, pretende divulgar el trabajo de investigación y desarrollo que se haga en esta Facultad y otras Facultades de Ingeniería del país.

Esperamos que esta sección pueda servir para aumentar los mecanismos de comunicación de la comunidad científico-tecnológica en el país. Consecuentes con lo anterior invitamos a investigadores de otras universidades para que usen este espacio para divulgar resultados que sean de interés para un sector amplio de la ingeniería.

Efficient Computation of Locally Monotonic Regression

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Este artículo apareció publicado en la revista "Signal Processing Letters" del IEEE en Septiembre de 1996
Artículo premiado por el Fondo de Reconocimiento de la Actividad Académica

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Resumen

So far, the applicability of locally monotonic regression has been limited by the high computational costs of the available algorithms that compute them. We present a powerful theoretical result about the nature of these regressions. As an application, we give an algorithm for the computation of lomo-3 regressions which reduces the complexity of the task, from exponential to polynomial.

I. INTRODUCTION

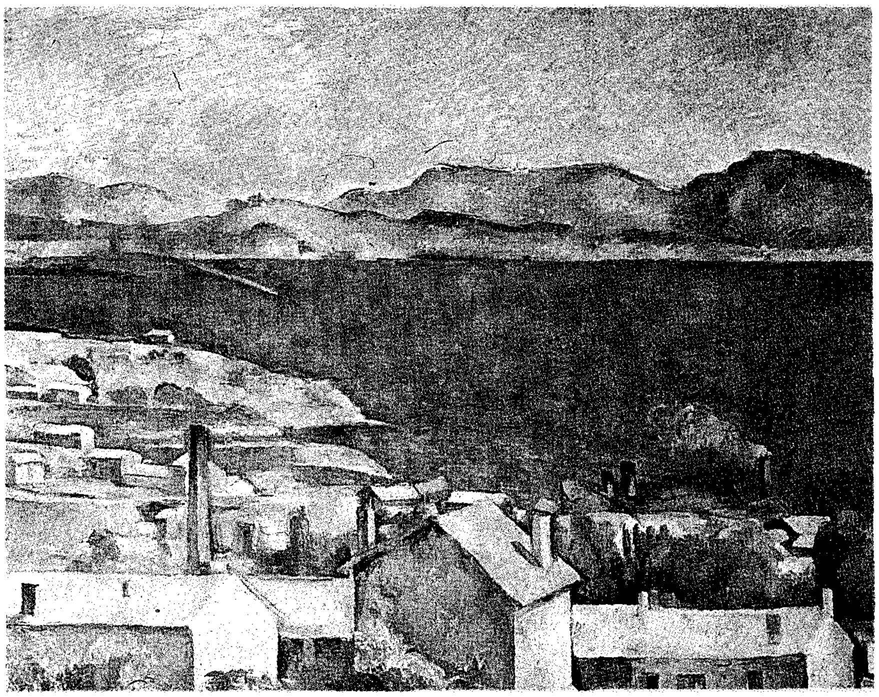
Locally Monotonic Regression (1) provides a way of smoothing signals under the smoothness criterion of local monotonicity, which sets a restriction on how often a signal may change trend (increasing to decreasing, or viceversa). Deterministically, locally monotonic (lomo, for short) regression provides signals that are locally monotonic and closest, under a given semimetric, to a given signal. Statistically, locally monotonic regression provides maximum likelihood estimators (2) of locally monotonic signals embedded in noise.

Lomo regression may well prove to be a useful smoothing tool; up to now, a drawback had been the extremely high computational costs for computing signals of reasonable lengths. Previous algorithms were combinatorial and had an exponential complexity.

Lomo regressions are obtained flatting segments of the signal being regressed. We show here that it is not necessary to flat segments of length larger than or equal to 2 $(\alpha-1)$, where α is the desired degree of local monotonicity. Using this fact, algorithms with polynomial complexity may be obtained. We present one such algorithm for $\alpha = 3$.

II. BASIC RESULTS

If n is a positive integer, a signal of length n is an element of $\boldsymbol{R}^n$, say $x = [x_1, ..., x_n]$. If α is a positive integer, $n \geq \alpha$, a signal of length n is said to be locally monotonic of degree a (or lomo-α) if each of its segments of length α is monotonic. The locally monotonic regressions of degree α of a signal x, are the lomo-α signals in $\mathbf{R}^n$ that are closest to x, according to a semimetric for $\mathbf{R}^n$. Here, we consider the Euclidean metric only.



The constant regressions of a signal x, are the constant signals of $\mathbf{R}^n$ that are closest to x. Under the Euclidean metric, constant regressions are unique and the value of the components of the constant regression of x is the average of its components. By *flatting* a segment of a signal x, we mean to replace the segment with its constant regression, obtaining a signal of the same length as x. The locally monotonic regressions of a signal can be obtained by flatting non-overlapping segments of the signal (1).

If $\mathbf{x} = [x_1, ..., x_n]$ is a signal of length n, the average of its components is denoted as $p(\mathbf{x}) = (x_1 + ... + x_n)/n$. Similarly, its constant Euclidean regression $[p(\mathbf{x}), ..., p(\mathbf{x})]$ is denoted as $\mathbf{p(x)}$. If, in addition, $\mathbf{y} = [y_1, ..., y_m]$ is a signal of length m, the concatenation $\mathbf{x}|\mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$ is the signal of length $n + m$ given by $[x_1, ..., x_n, y_1, ..., y_m]$. If $\mathbf{x}$ and $\mathbf{y}$ are signals of length n, we denote the Euclidean distance between them as $d(\mathbf{x}, \mathbf{y})$.

Lemma:

Let $\mathbf{x} = [x_1, ... x_n]$ be a signal of length n, let $\mathbf{a} = [a, ..., a]$ and $\mathbf{b} = [b, ..., b]$ be constant signals of length n. If $|p(\mathbf{x}) - a| < |p(\mathbf{x}) - b|$, then $d(\mathbf{x}, \mathbf{a}) < d(\mathbf{x}, \mathbf{b})$. That is, the closer the level of a constant signal is to the average of the components of a given signal, the closer the constant signal is to the given signal.

Proof:

$$d^2(\mathbf{x}, \mathbf{b}) - d^2(\mathbf{x}, \mathbf{a}) = \sum_{i=1}^{n} (x_i - b)^2 - \sum_{i=1}^{n} (x_i - a)^2$$

$$= \sum_{i=1}^{n} (x_i^2 - 2bx_i + b^2)$$

$$- \sum_{i=1}^{n} (x_i^2 - 2ax_i + a^2)$$

$$= n(b - p(\mathbf{x}))^2 - n(a - p(\mathbf{x}))^2$$

$$> 0.$$

Theorem:

In order to get the locally monotonic regressions of degree a of a signal, it is not necessary to flat segments of length larger than or equal to $2(\alpha - 1)$.

Proof:

Let n and a be positive integers with $n \geq \alpha$; let $\mathbf{x}$ be a signal of length n and let $\mathbf{s}$ be a lomo-a regression of $\mathbf{x}$. Let $\mathbf{s} = \mathbf{s}^1| ... |\mathbf{s}^k$ be the segmentation of $\mathbf{s}$ into (longest) constant segments and $\mathbf{x} = \mathbf{x}^1| ... |\mathbf{x}^k$ be the corresponding partition (not necessarily into constant segments) of $\mathbf{x}$. We show that each segment $\mathbf{x}^r$, $r \in /1, k/$, can be segmented into segments of length no larger than $2\alpha - 3$ whose constant regressions are the corresponding segments of $\mathbf{s}^r$.

Let $r \in /1, k/$, let m be the length of $\mathbf{x}^r$ and let $\mathbf{s}^r = [s_{l+1}, ..., s_{l+m}]$; then, $s_{l+1} = ... = s_{l+m} = p(\mathbf{x}^r)$. If $m < 2(\alpha - 1)$, there is nothing left to prove. Otherwise, if $m > 2(a - 1)$, let $\mathbf{z}^1 = [x_{l+1}, ..., x_{l+a-1}]$ be the initial segment

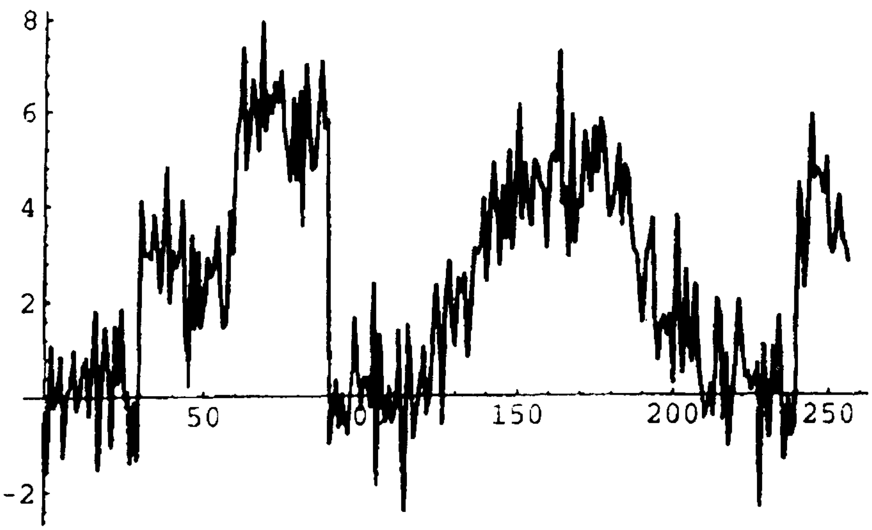


Fig. 1. A signal of length 256.

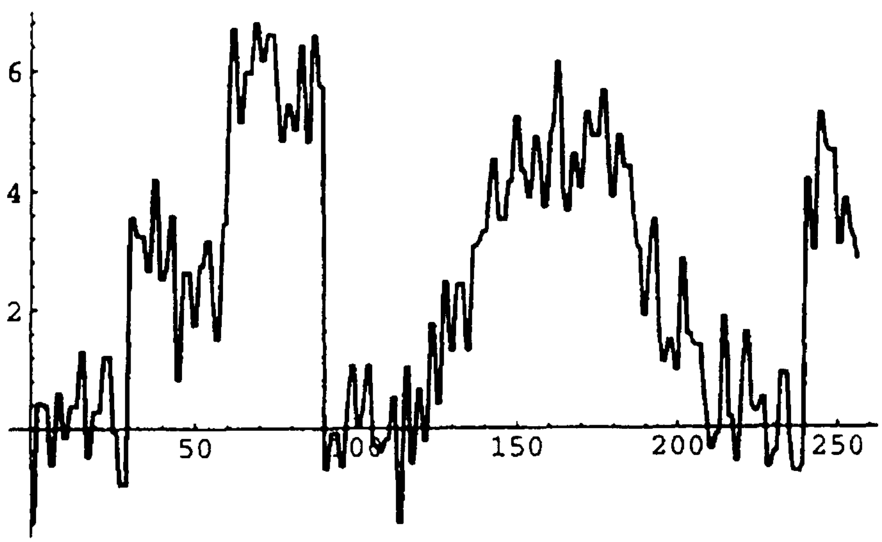


Fig. 2. A lomo-3 regression of the signal in Fig. 1.

of length $\alpha-1$ of $\mathbf{x}^r$ and $\mathbf{z}^2 = [x_{l+m-\alpha+2}, \ldots, x_{l+m}]$ be the ending segment of length $\alpha-1$ of $\mathbf{x}^r$. Also, let $\mathbf{z}^3 = [x_{l+\alpha}, \ldots, x_{l+m-\alpha+1}]$ be the intermediate remaining segment of $\mathbf{x}^r$; thus, $\mathbf{x}^r = \mathbf{z}^1 \mid \mathbf{z}^3 \mid \mathbf{z}^2$ and $\mathbf{z}^3$ is empty if $m = 2(\alpha - 1)$.
We claim that $p(\mathbf{z}^1) = p(\mathbf{x}^r)$ or, in other words, that the initial segment of $\mathbf{s}^r$ of length $\alpha-1$ is the constant regression of the corresponding segment $\mathbf{z}^1$ of $\mathbf{x}^r$. By contradiction, assume $p(\mathbf{z}^1) \neq p(\mathbf{x}^i)$; without loss of generality assume that $p(\mathbf{z}^1) < p(\mathbf{x}^i)$; using the Lemma above, it can be shown that the signal

$$\mathbf{s}^* = \mathbf{s}^1 \mid \ldots \mid \mathbf{s}^{r-1} \mid \mathbf{p}(\mathbf{z}^1) \mid [s_{l+\alpha}, \ldots, s_{l+m}] \mid \mathbf{s}^{r+1} \mid \ldots \mid \mathbf{s}^k$$

is closer to $\mathbf{x}$ than $\mathbf{s}$ is and therefore $\mathbf{s}^*$ is not lomo-a. Then, $[s_{l-\alpha+1}, \ldots, s_l]$ is nonconstant nondecreasing and $p(\mathbf{z}^1) < s_l < p(\mathbf{x}^i)$; then the signal,

$$\mathbf{s}^1 \mid \ldots \mid \mathbf{s}^{r-1} \mid [s_p, \ldots, s_l] \mid [s_{l+\alpha}, \ldots, s_{l+m}] \mid \mathbf{s}^{r+1} \mid \ldots \mid \mathbf{s}^k$$

where the segment $[s_p, \ldots, s_l]$ is of length $\alpha - 1$, is lomo-a and, using the Lemma, is closer to $\mathbf{x}$ than $\mathbf{s}$ is; this is a contradiction since $\mathbf{s}$ is a regression of $\mathbf{x}$. Then, $p(\mathbf{z}^1) = p(\mathbf{x}^r)$. Similarly, it can be shown that $p(\mathbf{z}^2) = p(\mathbf{x}^i)$; if $\mathbf{z}^3$ is empty, there is nothing left to prove. Otherwise, it remains to consider the segments of $\mathbf{z}^3$. Note that since $p(\mathbf{x}^i) = p(\mathbf{z}^1 \mid \mathbf{z}^3 \mid \mathbf{z}^2)$, $p(\mathbf{z}^1) = p(\mathbf{x}^i)$ and $p(\mathbf{z}^2) = p(\mathbf{x}^i)$, then $p(\mathbf{z}^3) = p(\mathbf{x}^i)$. Consider two cases: the case where the length of $\mathbf{z}^3$ is less than $2(\alpha - 1)$ and the case where it is larger than or equal to $2(\alpha - 1)$. In the first case we are done since $\mathbf{z}^1$, $\mathbf{z}^2$ and $\mathbf{z}^3$ are the segments of $\mathbf{x}^r$ that are being looked for. In the second case, expressing $\mathbf{z}^3$ as $\mathbf{z}^3 = \mathbf{y}^1 \mid \ldots \mid \mathbf{y}^q$, where the $\mathbf{y}^j$'s are segments of length larger than or equal to a − 1 and less than $2(\alpha - 1)$, from the Lemma (and knowing that $p(\mathbf{z}^1) = p(\mathbf{z}^2) = p(\mathbf{z}^3) = p(\mathbf{x}^r)$) it follows that for $j \in [1, q]$, $p(\mathbf{y}^j) = p(\mathbf{x}^r)$, otherwise a lomo-a signal closer to $\mathbf{x}$ can be found; thus, $\mathbf{z}^1, \mathbf{z}^2, \mathbf{y}^1, \ldots \mathbf{y}^q$ are the segments of $\mathbf{x}^r$ being looked for, and the proof is complete.

III. AN APPLICATION

Here, based on the Theorem, we give the main step for a recursive algorithm that computes lomo-3 regressions.

Let $x = [x_1, \ldots, x_n]$ be the input signal of length n, of which an output lomo-3 regression signal y is desired. Let m be the integer part of $(n + 1) / 2$.

Consider the following 8 ways of partitioning the signal $\mathbf{y}$ into three segments, $\mathbf{y} = \mathbf{y}^1 | \mathbf{y}^2 | \mathbf{y}^3$:

1. $y^1 = [y_1, \ldots, y_{m-1}]$,

   $y^2 = [x_m, x_{m+1}]$,

   $y^3 = [y_{m+2}, \ldots, y_L]$

2. $y^1 = [y_1, \ldots, y_{m-1}]$,

   $y^2 = [\overline{x_m, x_{m+1}}]$

   $y^3 = [y_{m+2}, \ldots, y_L]$

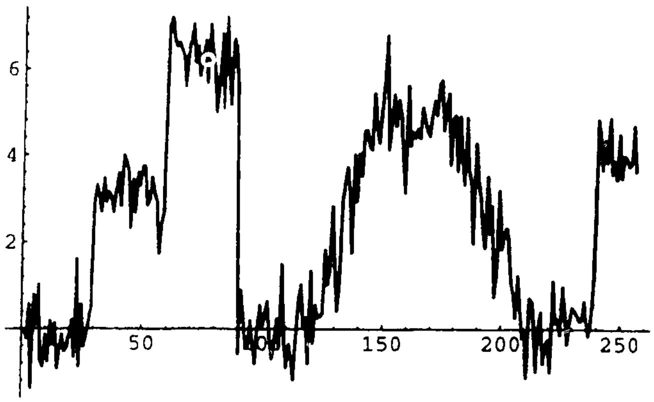


Fig. 3. A 256-pt signal.

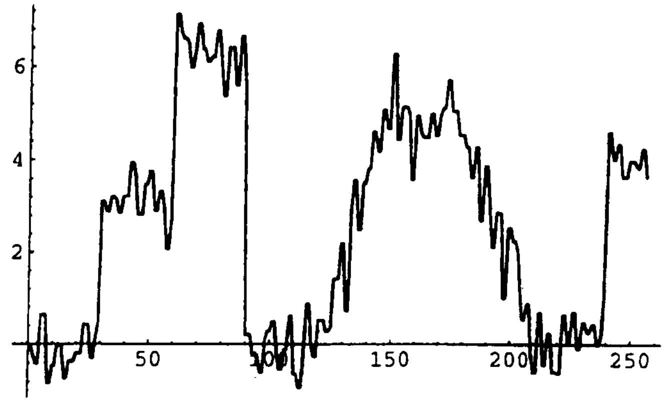


Fig. 4. A lomo-3 regression of the signal in Fig. 3.

3. $y^1 = [y_1, \ldots, y_{m-1}]$,

$y^2 = \overline{[x_m, x_{m+1}, x_{m+2}]}$,

$y^3 = [y_{m+3}, \ldots, y_L]$

4. $y^1 = [y_1, \ldots, y_m]$,

$y^2 = \overline{[x_{m+1}, x_{m+2}]}$,

$y^3 = [y_{m+3}, \ldots, y_L]$

5. $y^1 = [y_1, \ldots, y_{m-2}]$,

$y^2 = \overline{[x_{m-1}, x_m, x_{m+1}]}$,

$y^3 = [y_{m+2}, \ldots, y_L]$

6. $y^1 = [y_1, \ldots, y_{m-2}]$,

$y^2 = \overline{[x_{m-1}, x_m]}$,

$y^3 = [y_{m+1}, \ldots, y_L]$

7. $y^1 = [y_1, \ldots, y_m]$,

$y^2 = \overline{[x_{m+1}, x_{m+2}, x_{m+3}]}$,

$y^3 = [y_{m+4}, \ldots, y_L]$

8. $y^1 = [y_1, \ldots, y_{m-3}]$,

$y^2 = \overline{[x_{m-2}, x_{m-1}, x_m]}$

$y^3 = [y_{m+1}, \ldots, y_L]$

Cases 3 and 4 are considered only if $n > m+2$, *cases 5 and 6 only if* $m-1 > 1$, *case 7 only if* $n > m+3$ *and case 8 only if* $m-2 > 1$.

The algorithm proceeds recursively, calling itself with input signals $\mathbf{y}^1$ and $\mathbf{y}^3$ until the signal under consideration has a length less than 2. Finally, among the so obtained signals that turn out to be lomo-3, one that is closest to $\mathbf{x}$ is chosen.

We ran the algorithms on the 256-pt signals shown in Figures 1 and 3. The resulting smoothed signals are shown in Figures 2 and 4, respectively.

IV. CONCLUSION

An important result concerning lomo-a regressions has been presented. An algorithm for the computation of lomo-3 regressions has been described; its complexity is polynomial rather than exponential. We have knowledge of faster (Viterbi-type) algorithms (3) for computing lomo approximations with signals defined on a finite-length alphabet. Since the complexity of these algorithms grows with the square of the cardinality of the alphabet and since the problem for real valued signals is solvable, we think steps that reduce the complexity of algorithms that compute regressions, in contrast to digital regressions, are important. Lomo regression is a smoothing tool with applications in one-dimensional data analysis and in contrast-preserving image processing, for example.

REFERENCES

(1) A. Restrepo and A. C. Bovik, «Locally Monotonic Regression», *IEEE Trans. Signal Process.*, vol. 41, no. 9, pp. 2796-2810, 1993.

(2) A. Restrepo and A. C. Bovik, «On the Statistical Optimality of Locally Monotonic Regression», *IEEE Trans. Signal Process.*, vol. 42, no. 6, pp. 1553-1555, 1994.

(3) N.D. Sidiropoulos, «Fast Digital Locally Monotonic Regression», personal communication. 1995.